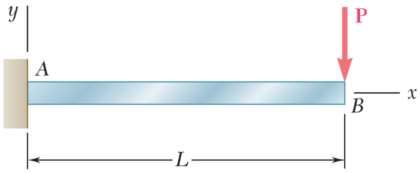


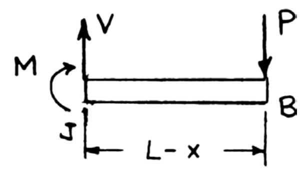
PROBLEM 9.1

For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.

SOLUTION

$$+\circlearrowleft \Sigma M_J = 0: \quad -M - P(L-x) = 0$$

$$M = -P(L-x)$$



$$EI\frac{d^2y}{dx^2} = -P(L-x) = -PL + Px$$

$$EI\frac{dy}{dx} = -PLx + \frac{1}{2}Px^2 + C_1$$

$$\left[x = 0, \ \frac{dy}{dx} = 0\right]: \qquad 0 = -0 + 0 + C_1 \qquad C_1 = 0$$

$$EIy = -\frac{1}{2}PLx^2 + \frac{1}{6}Px^3 + C_1x + C_2$$

$$[x = 0, \ y = 0]: \qquad 0 = -0 + 0 + 0 + C_2 \qquad C_2 = 0$$

(*a*) Elastic curve.

$$y = -\frac{Px^2}{6EI}(3L - x) \blacktriangleleft$$

$$\frac{dy}{dx} = -\frac{Px}{2EI}(2L - x)$$

(*b*) y at $x = L$.

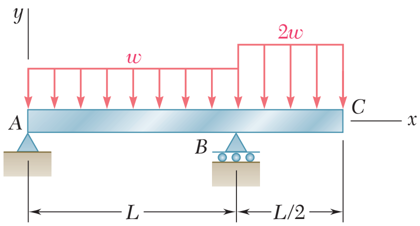
$$y_B = -\frac{PL^2}{6EI}(3L - L) = -\frac{PL^3}{3EI}$$

$$y_B = \frac{PL^3}{3EI} \downarrow \blacktriangleleft$$

(*c*) $\frac{dy}{dx}$ at $x = L$.

$$\left.\frac{dy}{dx}\right|_B = -\frac{PL}{2EI}(2L - L) = -\frac{PL^2}{2EI}$$

$$\theta_B = \frac{PL^2}{2EI} \searrow \blacktriangleleft$$



PROBLEM 9.8

For the beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the slope at *A*, (*c*) the slope at *B*.

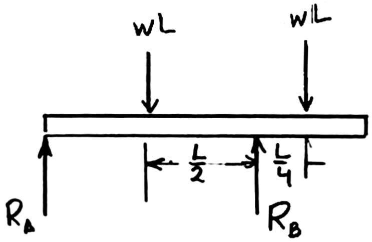
SOLUTION

Using free body *ABC*,

$[x = 0, y = 0] \quad [x = L, y = 0]$

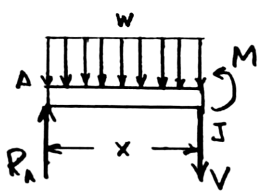
$$+\circlearrowleft \Sigma M_B = 0: \quad -R_A L + (wL)\left(\frac{L}{2}\right) - (wL)\left(\frac{L}{4}\right) = 0$$

$$R_A = \frac{1}{4} wL$$



For portion *AB*, $(0 < x < L)$

$$+\circlearrowleft \Sigma M_J = 0: \quad M - R_A x + (wx)\left(\frac{x}{2}\right) = 0$$



$$M = \frac{1}{4} wLx - \frac{1}{2} wx^2$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{4} wLx - \frac{1}{2} wx^2$$

$$EI\frac{dy}{dx} = \frac{1}{8} wLx^2 - \frac{1}{6} wx^3 + C_1$$

$$EIy = \frac{1}{24} wLx^3 - \frac{1}{24} wx^4 + C_1 x + C_2$$

$[x = 0, y = 0]: \quad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$

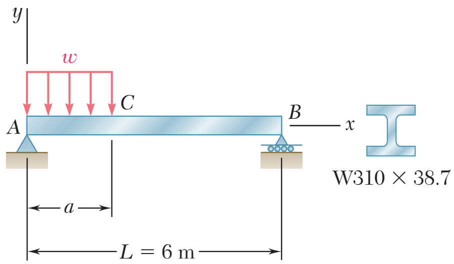
$[x = L, y = 0]: \quad 0 = \frac{1}{24} wL^4 - \frac{1}{24} wL^4 + C_1 L + 0 = 0 \qquad C_1 = 0$

(*a*) Elastic curve $(0 \le x \le L)$. $\qquad y = \frac{w}{24EI}(Lx^3 - x^4)$ ◀

$$\frac{dy}{dx} = \frac{w}{24EI}(3Lx^2 - 4x^3)$$

(*b*) $\frac{dy}{dx}$ at $x = 0$. $\qquad \left.\frac{dy}{dx}\right|_A = 0 \qquad \theta_A = 0$ ◀

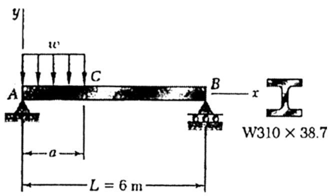
(*c*) $\frac{dy}{dx}$ at $x = L$. $\qquad \left.\frac{dy}{dx}\right|_B = -\frac{wL^3}{24EI} \qquad \theta_B = \frac{wL^3}{24EI}$ ↘ ◀



PROBLEM 9.15

For the beam and loading shown, knowing that $a = 2$ m, $w = 50$ kN/m, and $E = 200$ GPa, determine (*a*) the slope at support *A*, (*b*) the deflection at point *C*.

SOLUTION



Using ACB as a free body and noting that $L = 3a$,

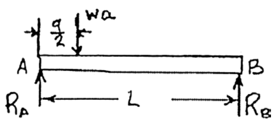
$$+\circlearrowleft \Sigma M_A = 0: \quad R_B L - (wa)\left(\frac{a}{2}\right) = 0$$

$[x = 0,\ y = 0]$ $\quad$ $[x = L,\ y = 0]$

$$R_B = (wa)\frac{a}{2L} = \frac{1}{6}wa$$

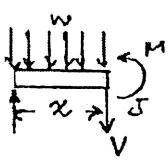
$[x = a,\ y = y]$

$$\left[x = a,\ \frac{dy}{dx} = \frac{dy}{dx}\right]$$



$$+\uparrow \Sigma F_y = 0: \quad R_A + R_B - wa = 0 \qquad R_A = \frac{5}{6}wa$$

$0 \le x \le a$



$$+\circlearrowleft \Sigma M_J = 0:$$

$$M - R_A x + (wx)\left(\frac{x}{2}\right) = 0$$

$$M = R_A x - \frac{1}{2}wx^2$$

$$EI\frac{d^2y}{dx^2} = R_A x - \frac{1}{2}wx^2$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{6}wx^3 + C_1$$

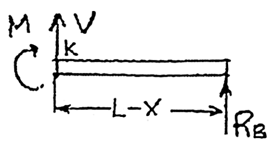
$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}wx^4 + C_1 x + C_2$$

$$[x = 0,\ y = 0] \quad 0 = 0 - 0 + 0 + C_2 \quad C_2 = 0$$

$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}wx^4 + C_1 x$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{6}wx^3 + C_1$$

$a \le x \le L$



$$+\circlearrowleft \Sigma M_K = 0:$$

$$-M + R_B(L - x) = 0$$

$$M = R_B(L - x)$$

$$EI\frac{d^2y}{dx^2} = R_B(L - x)$$

$$EI\frac{dy}{dx} = -\frac{1}{2}R_B(L - x)^2 + C_3$$

$$EIy = \frac{1}{6}R_B(L - x)^3 + C_3 x + C_4$$

$$[x = L,\ y = 0] \quad 0 = 0 + C_3 L + C_4 \qquad C_4 = -C_3 L$$

$$EIy = \frac{1}{6}R_B(L - x)^2 - C_3(L - x)$$

$$EI\frac{dy}{dx} = -\frac{1}{2}R_B(L - x)^2 + C_3$$

$$\left[x = a,\ \frac{dy}{dx} = \frac{dy}{dx}\right] \qquad \frac{1}{2}R_A a^2 - \frac{1}{6}wa^3 + C_1 = -\frac{1}{2}R_B(2a)^2 + C_3$$

$$C_3 = C_1 + \frac{1}{2}R_A a^2 - \frac{1}{6}wa^3 + \frac{1}{2}R_B(2a)^2 = C_1 + \frac{7}{12}wa^3$$

$$[x = a,\ y = y] \qquad \frac{1}{6}R_A a^3 - \frac{1}{24}wa^4 + C_1 a = \frac{1}{6}R_B(2a)^3 - \left(C_1 + \frac{7}{12}wa^3\right)(2a)$$

$$3C_1 a = -\frac{1}{6}R_A a^3 + \frac{1}{24}wa^4 + \frac{1}{6}R_B(2a)^3 - \frac{7}{12}wa^2(2a) = -\frac{25}{24}wa^4$$

$$C_1 = -\frac{25}{72}wa^3$$

For $0 \le x \le a$, $EIy = \frac{5}{36}wax^3 - \frac{1}{24}wx^4 - \frac{25}{72}wa^3 x$

$$EI\frac{dy}{dx} = \frac{5}{12}wax^2 - \frac{1}{6}wx^3 - \frac{25}{72}wa^3$$

<u>Data:</u> $w = 50 \times 10^3$ N/m, $a = 2$ m, $E = 200 \times 10^9$ Pa

$I = 84.9 \times 10^6\ \text{mm}^4 = 84.9 \times 10^{-6}\ \text{m}^4$, $EI = 16.98 \times 10^6\ \text{N} \cdot \text{m}^2$

(a) <u>Slope at $x = 0$</u>.

$$16.98 \times 10^6 \left.\frac{dy}{dx}\right|_A = 0 - 0 - \frac{25}{72}(50 \times 10^3)(2)^3$$

$$\left.\frac{dy}{dx}\right|_A = \theta_A = -8.18 \times 10^{-3}$$

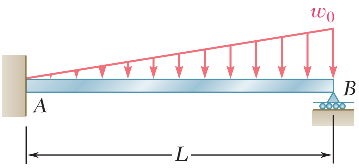
$\theta_A = 8.18 \times 10^{-3}$ rad ◀

(b) <u>Deflection at $x = 2$ m</u>.

$$EIy_C = \frac{5}{36}wa^4 - \frac{1}{24}wa^4 - \frac{25}{72}wa^4 = -\frac{1}{4}wa^4$$

$$16.98 \times 10^6 y_C = -\frac{1}{4}(50 \times 10^3)(2)^4 \qquad y_C = -11.78 \times 10^{-3}\ \text{m}$$

$y_C = 11.78$ mm ↓ ◀



$[x = 0,\ y = 0]$ $[x = L,\ y = 0]$

$\left[x = 0,\ \frac{dy}{dx} = 0\right]$

PROBLEM 9.21

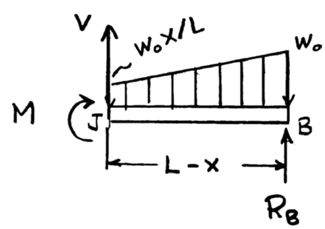
For the beam and loading shown, determine the reaction at the roller support.

SOLUTION

Reactions are statically indeterminate.

Boundary conditions are shown above.

Using free body *JB*,



$$+\circlearrowleft \Sigma M_J = 0:\ -M + R_B(L - x) + \frac{1}{2}w_0(L - x)\frac{2}{3}(L - x) + \frac{1}{2}\frac{w_0 x}{L}(L - x)\frac{1}{3}(L - x) = 0$$

$$M = R_B(L - x) - \frac{w_0}{6L}[2L(L - x)^2 + x(L - x)^2]$$

$$= R_B(L - x) - \frac{w_0}{6L}[2L^3 - 4L^2x + 2Lx^2 + xL^2 - 2Lx^2 + x^3]$$

$$= R_B(L - x) - \frac{w_0}{6L}(x^3 - 3L^2x + 2L^3)$$

$$EI\frac{d^2y}{dx^2} = R_B(L - x) - \frac{w_0}{6L}(x^3 - 3L^2x + 2L^3)$$

$$EI\frac{dy}{dx} = R_B\left(Lx - \frac{1}{2}x^2\right) - \frac{w_0}{6L}\left(\frac{1}{4}x^4 - \frac{3}{2}L^2x^2 + 2L^3x\right) + C_1$$

$$EIy = R_B\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) - \frac{w_0}{6L}\left(\frac{1}{20}x^5 - \frac{1}{2}L^2x^3 + L^3x^2\right) + C_1x + C_2$$

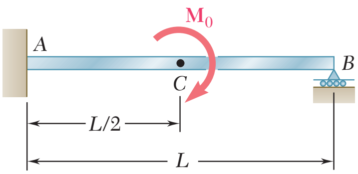
$$[x = 0,\ y = 0] \quad \rightarrow C_2 = 0$$

$$\left[x = 0,\ \frac{dy}{dx} = 0\right] \rightarrow C_1 = 0$$

$$[x = L,\ y = 0] \quad 0 = R_BL^3\left(\frac{1}{2} - \frac{1}{6}\right) - \frac{w_0L^4}{6}\left(\frac{1}{20} - \frac{1}{2} + 1\right)$$

$$\frac{1}{3}R_B = \left(\frac{1}{6}\right)\left(\frac{11}{20}\right)w_0L$$

$$R_B = \frac{11}{40}w_0L \uparrow \blacktriangleleft$$



$[x = 0,\ y = 0]$ $[x = L,\ y = 0]$

$\left[x = 0,\ \dfrac{dy}{dx} = 0\right]$

PROBLEM 9.26

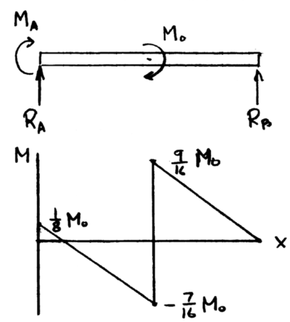
Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

SOLUTION

Reactions are statically indeterminate.

$$+\uparrow \Sigma F_y = 0: \quad R_A + R_B = 0 \qquad R_A = -R_B$$

$$+\circlearrowleft \Sigma M_A = 0: \quad -M_A - M_0 + R_B L = 0 \qquad M_A = R_B L - M_0$$



$\underline{0 < x < \dfrac{L}{2}}$

$$M = R_B x + M_A = -M_0 + R_B L - R_B x$$

$$EI\frac{d^2y}{dx^2} = -M_0 + R_B(L - x)$$

$$EI\frac{dy}{dx} = -M_0 x + R_B\left(Lx - \frac{1}{2}x^2\right) + C_1$$

$$EIy = -\frac{1}{2}M_0 x^2 + R_B\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + C_1 x + C_2$$

$\underline{\dfrac{L}{2} < x < L}$

$$M = R_B(L - x)$$

$$EI\frac{d^2y}{dx^2} = R_B(L - x)$$

$$EI\frac{dy}{dx} = R_B\left(Lx - \frac{1}{2}x^2\right) + C_3$$

$$EIy = R_B\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + C_3 x + C_4$$

$$\left[x = 0,\ \frac{dy}{dx} = 0\right] \quad 0 + 0 + C_1 = 0 \qquad C_1 = 0$$

$$[x = 0,\ y = 0] \quad 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$$

$$\left[x = \frac{L}{2},\ \frac{dy}{dx} = \frac{dy}{dx}\right]$$

$$-M_0\frac{L}{2} + \cancel{R_B\left(\frac{1}{2}L^2 - \frac{1}{6}L^2\right)} = \cancel{R_B\left(\frac{1}{2}L^2 - \frac{1}{6}L^2\right)} + C_3 \qquad C_3 = -\frac{M_0L}{2}$$

$$\left[x = \frac{L}{2},\ y = y\right]$$

$$-\frac{1}{2}M_0\left(\frac{L}{2}\right)^2 + \cancel{R_B\left(\frac{1}{8}L^3 - \frac{1}{48}L^3\right)} = \cancel{R_B\left(\frac{1}{8}L^3 - \frac{1}{48}L^3\right)} + C_3\frac{L}{2} + C_4$$

$$C_4 = -\frac{1}{8}M_0L^2 - \frac{1}{2}C_3L$$

$$= \left(-\frac{1}{8} + \frac{1}{4}\right)M_0L^2 = \frac{1}{8}M_0L^2$$

$$[x = L,\ y = 0]$$

$$R_B\left(\frac{1}{2}L^3 - \frac{1}{6}L^3\right) + \frac{M_0L}{2}L + \frac{1}{8}M_0L^2 = 0$$

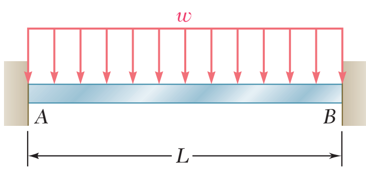
$$\left(\frac{1}{2} - \frac{1}{6}\right)R_BL^3 = \left(\frac{1}{2} - \frac{1}{8}\right)M_0L^2 \quad \frac{1}{3}R_B = \frac{3}{8}\frac{M_0}{L}$$

$$R_B = \frac{9}{8}\frac{M_0}{L}\uparrow \blacktriangleleft$$

$$M_A = \frac{9}{8}M_0 - M_0 = \frac{1}{8}M_0 \qquad M_A = \frac{1}{8}M_0 \blacktriangleleft$$

$$M_{C^-} = -M_0 + \frac{9}{8}\frac{M_0}{L} = -\frac{7}{16}M_0 \qquad M_{C^-} = -\frac{7}{16}M_0 \blacktriangleleft$$

$$M_{C^+} = R_B\left(L - \frac{L}{2}\right) = \frac{9}{8}\frac{M_0}{L}\left(\frac{L}{2}\right) = \frac{9}{16}M_0 \qquad M_{C^+} = \frac{9}{16}M_0 \blacktriangleleft$$



$[x = 0,\ y = 0]$ $[x = L,\ y = 0]$

$\left[x = 0,\ \dfrac{dy}{dx} = 0\right]$ $\left[x = L,\ \dfrac{dy}{dx} = 0\right]$

PROBLEM 9.33

Determine the reaction at A and draw the bending moment diagram for the beam and loading shown.

SOLUTION

Reactions are statically indeterminate.

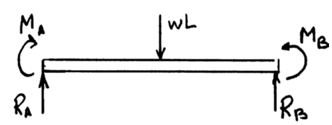
By symmetry, $R_B = R_A; \quad M_B = M_A$

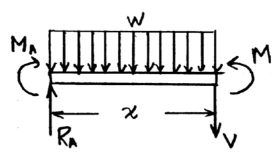
$$\frac{dy}{dx} = 0 \quad \text{at} \quad x = \frac{L}{2}$$

$$+\uparrow \Sigma F_y = 0: \quad R_A + R_B - wL = 0$$

$$R_B = R_A = \frac{1}{2} wL \uparrow \blacktriangleleft$$

Over entire beam, $M = M_A + R_A x - \dfrac{1}{2} w x^2$





$$EI \frac{d^2 y}{dx^2} = M_A + \frac{1}{2} wLx - \frac{1}{2} w x^2$$

$$EI \frac{dy}{dx} = M_A x + \frac{1}{4} wLx^2 - \frac{1}{6} w x^3 + C_1$$

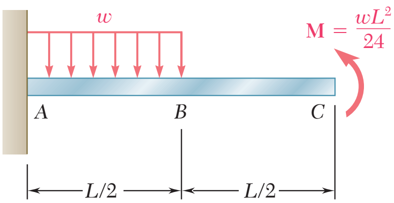
$$\left[x = 0,\ \frac{dy}{dx} = 0\right] \quad 0 + 0 - 0 + C_1 = 0 \qquad C_1 = 0$$

$$\left[x = \frac{L}{2},\ \frac{dy}{dx} = 0\right] \qquad \frac{1}{2} M_A L + \frac{1}{16} wL^3 - \frac{1}{48} wL^3 + 0 = 0$$

$$M_A = -\frac{1}{12} wL^2 \circlearrowleft \blacktriangleleft$$

$$M = -\frac{1}{12} wL^2 + \frac{1}{2} wLx - \frac{1}{2} w x^2$$

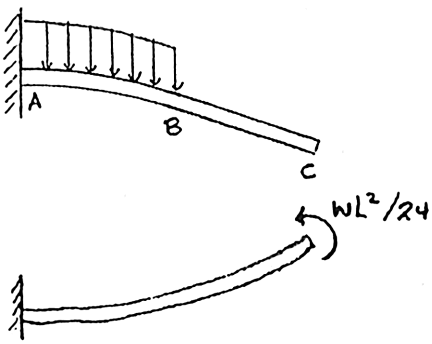
$$M = w[6x(L - x) - L^2]/12 \blacktriangleleft$$



PROBLEM 9.65

For the cantilever beam and loading shown, determine the slope and deflection at the free end.

SOLUTION



Loading I: Downward distributed load w applied to portion AB.

Case 2 of Appendix D applied to portion AB.

$$\theta'_B = -\frac{w(L/2)^3}{6EI} = -\frac{1}{48}\frac{wL^2}{EI}$$

$$y'_B = -\frac{w(L/2)^4}{8EI} = -\frac{1}{128}\frac{wL^4}{EI}$$

Portion BC remains straight.

$$\theta'_C = \theta'_B = -\frac{1}{48}\frac{wL^3}{EI}$$

$$y'_C = y'_B + \left(\frac{L}{2}\right)\theta'_B = -\frac{1}{128}\frac{wL^4}{EI} - \frac{1}{96}\frac{wL^4}{EI} = -\frac{7}{384}\frac{wL^4}{EI}$$

Loading II: Counterclockwise couple $\frac{wL^2}{24}$ applied at C.

Case 3 of Appendix D.

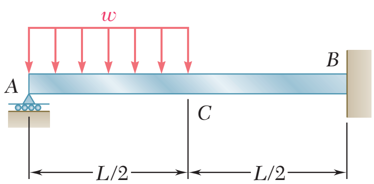
$$\theta''_C = \frac{(wL^2/24)L}{EI} = \frac{1}{24}\frac{wL^3}{EI}$$

$$y''_C = \frac{(wL^2/24)L^2}{2EI} = \frac{1}{48}\frac{wL^4}{EI}$$

By superposition,

$$\theta_C = \theta'_C + \theta''_C = -\frac{1}{48}\frac{wL^3}{EI} + \frac{1}{24}\frac{wL^3}{EI} \qquad \theta_C = \frac{1}{48}\frac{wL^3}{EI} \measuredangle \blacktriangleleft$$

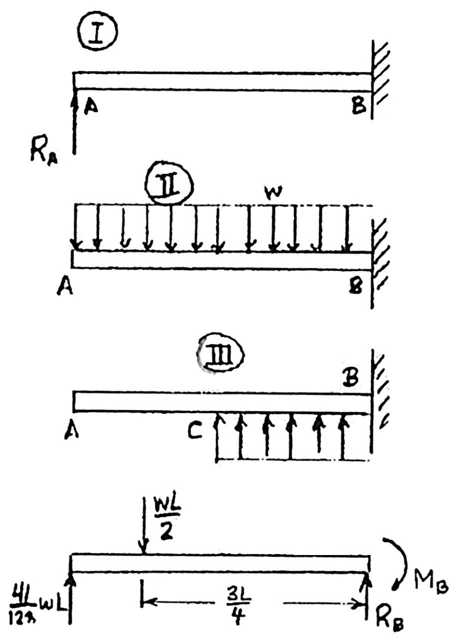
$$y_C = y'_C + y''_C = -\frac{7}{384}\frac{wL^4}{EI} + \frac{1}{48}\frac{wL^4}{EI} \qquad y_C = \frac{1}{384}\frac{wL^4}{EI} \uparrow \blacktriangleleft$$



PROBLEM 9.80

For the uniform beam shown, determine (*a*) the reaction at *A*, (*b*) the reaction at *B*.

SOLUTION



Beam is indeterminate to first degree. Consider R_A as redundant and replace the given loading by loadings I, II, and III.

Loading I: Case 1 of Appendix *D*.

$$(y_A)_I = \frac{R_A L^3}{3EI}$$

Loading II: Case 2 of Appendix *D*.

$$(y_A)_{II} = -\frac{wL^4}{8EI}$$

Loading III: Case 2 of Appendix *D* (portion *CB*).

$$(\theta_C)_{III} = -\frac{w(L/2)^3}{6EI} = -\frac{1}{48}\frac{wL^3}{EI}$$

$$(y_C)_{III} = \frac{w(L/2)^4}{8EI} = \frac{1}{128}\frac{wL^4}{EI}$$

Portion *AC* remains straight.

$$(y_A)_{III} = (y_C)_{III} + \frac{L}{2}(\theta_C)_{III} = \frac{7}{384}\frac{wL^4}{EI}$$

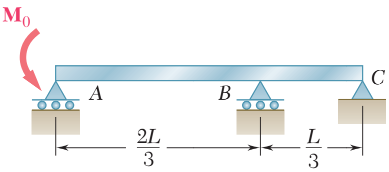
Superposition and constraint: $y_A = (y_A)_I + (y_A)_{II} + (y_A)_{III} = 0$

(*a*) $$\frac{1}{3}\frac{R_A L^3}{3EI} - \frac{1}{8}\frac{wL^4}{EI} + \frac{7}{384}\frac{wL^4}{EI} = \frac{1}{3}\frac{R_A L^3}{EI} - \frac{41}{384}\frac{wL^4}{EI} = 0$$ $R_A = \frac{41}{128}wL \uparrow$ ◀

Statics:

(*b*) $$+\uparrow \Sigma F_y = 0: \quad \frac{41}{128}wL - \frac{1}{2}wL + R_B = 0$$ $R_B = \frac{23}{128}wL \uparrow$ ◀

$$+\circlearrowleft \Sigma M_B = 0: \quad -\left(\frac{41}{128}wL\right)L - \left(\frac{1}{2}wL\right)\left(\frac{3L}{4}\right) - M_B = 0$$ $M_B = \frac{7}{128}wL^2 \circlearrowright$ ◀

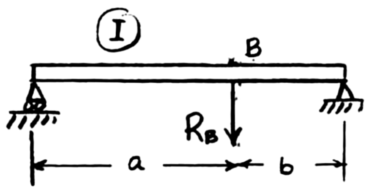


PROBLEM 9.82

For the uniform beam shown, determine the reaction at each of the three supports.

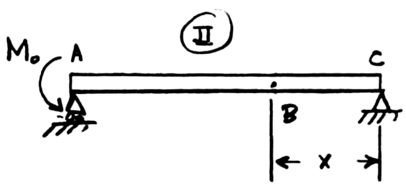
SOLUTION

Beam is statically indeterminate to first degree. Consider R_B to be the redundant reaction, and replace the loading by loading I and II.



<u>Loading I</u>: Case 5 of Appendix *D*.

$$(y_B)_I = -\frac{R_B a^2 b^2}{3EIL} = -\frac{R_B(2L/3)^2(L/3)^2}{3EIL} = -\frac{4}{243}\frac{R_B L^3}{EI}$$



<u>Loading II</u>: Case 7 of Appendix *D*.

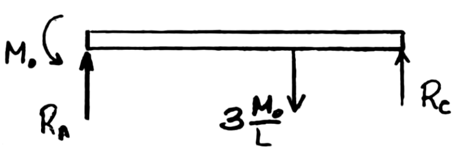
$$(y_B)_{II} = -\frac{M_0}{6EIL} - (x^3 - L^2x) = -\frac{M_0}{6EIL}\left[\left(\frac{L}{3}\right)^3 - L^2\left(\frac{L}{3}\right)\right]$$

$$= \frac{4}{81}\frac{M_0 L^2}{EI}$$

Superposition and constraint:

$$y_B = (y_B)_I + (y_B)_{II} = 0$$

$$-\frac{4}{243}\frac{R_B L^3}{EI} + \frac{4}{81}\frac{M_0 L^2}{EI} = 0 \qquad R_B = 3\frac{M_0}{L}\downarrow \blacktriangleleft$$



Statics:

$$+\circlearrowleft \Sigma M_C = 0: \quad -R_A L + M_0 + 3\frac{M_0}{L}\cdot\frac{L}{3} = 0 \qquad R_A = 2\frac{M_0}{L}\uparrow \blacktriangleleft$$

$$+\uparrow \Sigma F_y = 0: \quad 2\frac{M_0}{L} - 3\frac{M_0}{L} + R_C = 0 \qquad R_C = \frac{M_0}{L}\uparrow \blacktriangleleft$$